Technical Appendix for "Gross Migration, Housing and Urban Population Dynamics"*

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December 2, 2014

Abstract

This document derives and proves claims made in the main text. Note that some of the notation differs from the main text, but is internally consistent.

*Any views expressed herein are those of the authors and do not necessarily represent those of the Federal Reserve Bank of Chicago or the Federal Reserve System.

1 Static Migration Economy

1.1 Planner's problem

$$\max\left\{\ln\left(C\right)-\sum_{s,x}\left[H\left(\frac{a\left(s,x\right)}{x}\right)^{2}x+\left(-\psi_{1}\frac{l\left(s,x\right)}{x}+\psi_{2}\left(\frac{l\left(s,x\right)}{x}\right)^{2}\right)x\right]\mu\left(s,x\right)-\tau\Lambda\right\}$$

subject to:

$$p\left(s,x\right)\leq x+a\left(s,x\right)+\Lambda x-l\left(s,x\right)$$

$$\sum_{s,x}\left[a\left(s,x\right)+\Lambda x\right]\mu\left(s,x\right)\leq\sum_{s,x}l\left(s,x\right)\mu\left(s,x\right)$$

$$a\left(s,x\right)\geq0$$

$$l\left(s,x\right)\geq0$$

$$C\leq\sum_{s,x}sp\left(s,x\right)^{\theta}\mu\left(s,x\right)$$

FOC's:

$$-\tau+\sum_{s,x}\lambda\xi\left(s,x\right)x\mu\left(s,x\right)-\lambda\eta\leq0\,(\text{with }=\text{ if }\Lambda>0)$$

$$-2H\frac{a\left(s,x\right)}{x}+\lambda\xi\left(s,x\right)-\lambda\eta\leq0\,(\text{with }=\text{ if }a(s,x)>0)$$

$$\left[\psi_{1}-2\psi_{2}\frac{l\left(s,x\right)}{x}\right]-\lambda\xi\left(s,x\right)+\lambda\eta\leq0\,(\text{with }=\text{ if }l(s,x)>0)$$

$$\xi\left(s,x\right)=s\theta p\left(s,x\right)^{\theta-1}$$

$$C=\sum_{s,x}sp\left(s,x\right)^{\theta}\mu\left(s,x\right)$$

For the model to be consistent with the gross flows data we need that $l(s,x)>0$ and $a\left(s,x\right)>0$ almost everywhere. The reason we need that $l(s,x)>0$ almost everywhere is that leaving rates are always positive in the data. The reason we need that $a(s,x)>0$ almost everywhere is that otherwise we would have sets of positive measure in which total arrival rates are equal to Λ, which is independent of net population changes $\frac{p(s,x)-x}{x}$ and inconsistent with the data.

Suppose that $a(s,x)>0$ and $l(s,x)>0$ almost everywhere.

Then,

$$2H\frac{a\left(s,x\right)}{x}=\psi_{1}-2\psi_{2}\frac{l\left(s,x\right)}{x}$$

i.e.

$$\frac{l\left(s,x\right)}{x}=\frac{1}{2}\frac{\psi_{1}}{\psi_{2}}-\frac{H}{\psi_{2}}\frac{a\left(s,x\right)}{x}$$

Observe that:

$$\frac{p(s,x)-x}{x} = \frac{a(s,x)}{x} + \Lambda - \frac{l(s,x)}{x}$$

Then,

$$\begin{aligned}\frac{a(s,x)}{x} &= \frac{p(s,x)-x}{x} - \Lambda + \frac{l(s,x)}{x} \\ &= \frac{p(s,x)-x}{x} - \Lambda + \frac{1}{2}\frac{\psi_1}{\psi_2} - \frac{H}{\psi_2}\frac{a(s,x)}{x}\end{aligned}$$

It follows that

$$\frac{a(s,x)}{x} = \frac{\psi_2}{\psi_2+H}\left(\frac{p(s,x)-x}{x}\right) + \frac{\psi_2}{\psi_2+H}\left(\frac{1}{2}\frac{\psi_1}{\psi_2} - \Lambda\right)$$

It follows that the total arrival rate is given by:

$$\frac{a(s,x)}{x} + \Lambda = \frac{\psi_2}{\psi_2+H}\left(\frac{p(s,x)-x}{x}\right) + \frac{\psi_2}{\psi_2+H}\left(\frac{1}{2}\frac{\psi_1}{\psi_2} - \Lambda\right) + \Lambda,$$

which is a linear function of the net population change $(p-x)/x$. Moreover, the linear coefficient is:

$$0 < \frac{\psi_2}{\psi_2+H} < 1.$$

Also, the leaving rate is given by:

$$\begin{aligned}\frac{l(s,x)}{x} &= \frac{a(s,x)}{x} + \Lambda - \left(\frac{p(s,x)-x}{x}\right) \\ &= \left(-\frac{H}{\psi_2+H}\right)\left(\frac{p(s,x)-x}{x}\right) + \frac{\psi_2}{\psi_2+H}\left(\frac{1}{2}\frac{\psi_1}{\psi_2} - \Lambda\right) + \Lambda\end{aligned}$$

which also is a linear function of the net population change $(p-x)/x$, with the linear coefficient given by

$$-1 < -\frac{H}{\psi_2+H} < 0.$$

Since the empirical linear coefficient $\frac{\psi_2}{\psi_2+H}$ is a number between 0 and 1, we need both that $H > 0$ and that $\psi_2 > 0$.

Assume for the moment that $\psi_1 \leq 0$. Then,

$$\frac{a(s,x)}{x} = \frac{\psi_2}{\psi_2+H}\left(\frac{p(s,x)-x}{x}\right) + \frac{\psi_2}{\psi_2+H}\left(\frac{1}{2}\frac{\psi_1}{\psi_2} - \Lambda\right)$$

becomes negative when net population changes $\frac{p(s,x)-x}{x}$ are non-positive, which is a contradiction. Thus we need that $\psi_1 > 0$ for the model to be consistent with the empirical adjustments estimated.

1.2 Alternative specifications for moving costs:

1) Suppose that $H = 0$.

Then, the FOC's become

$$\lambda\xi(s,x) - \lambda\eta \leq 0 \,(\text{with } = \text{ if } a(s,x) > 0)$$

$$\psi_1 - 2\psi_2 \frac{l(s,x)}{x} \leq \lambda\xi(s,x) - \lambda\eta \,(\text{with } = \text{ if } l(s,x) > 0)$$

Hence:

$$\frac{\psi_1}{2\psi_2} \leq \frac{l(s,x)}{x}.$$

Suppose that $\frac{\psi_1}{2\psi_2} < \frac{l(s,x)}{x}$ and that $a(s,x) > 0$.

Hence,

$$\psi_1 - 2\psi_2 \frac{l(s,x)}{x} = \lambda\xi(s,x) - \lambda\eta = 0.$$

A contradiction. Hence $\frac{\psi_1}{2\psi_2} < \frac{l(s,x)}{x}$ implies that $a(s,x) = 0$

i) Suppose that $\frac{p(s,x)-x}{x} - \Lambda + \frac{\psi_1}{2\psi_2} > 0$. Then,

$$\frac{a(s,x)}{x} = \frac{p(s,x)-x}{x} - \Lambda + \frac{l(s,x)}{x} \geq \frac{p(s,x)-x}{x} - \Lambda + \frac{\psi_1}{2\psi_2} > 0.$$

It follows that

$$\begin{aligned} \frac{l(s,x)}{x} &= \frac{\psi_1}{2\psi_2} \\ \frac{a(s,x)}{x} &= \frac{p(s,x)-x}{x} - \Lambda + \frac{\psi_1}{2\psi_2} \end{aligned}$$

ii) Suppose that $\frac{p(s,x)-x}{x} - \Lambda + \frac{\psi_1}{2\psi_2} \leq 0$.

Suppose that $a(s,x) > 0$. Then, $\frac{\psi_1}{2\psi_2} = \frac{l(s,x)}{x}$ and

$$\frac{a(s,x)}{x} = \frac{p(s,x)-x}{x} - \Lambda + \frac{l(s,x)}{x} = \frac{p(s,x)-x}{x} - \Lambda + \frac{\psi_1}{2\psi_2} \leq 0$$

A contradiction.

Hence

$$\begin{aligned} \frac{a(s,x)}{x} &= 0 \\ \frac{l(s,x)}{x} &= \Lambda - \left(\frac{p(s,x)-x}{x}\right) \geq \frac{\psi_1}{2\psi_2} \end{aligned}$$

iii) From i) and ii) we then have that

$$\begin{aligned}
\frac{a(s,x)}{x} &= \max\left\{\frac{p(s,x)-x}{x} - \Lambda + \frac{\psi_1}{2\psi_2}, 0\right\} \\
\frac{l(s,x)}{x} &= \frac{a(s,x)}{x} - \left(\frac{p(s,x)-x}{x}\right) + \Lambda \\
&= \max\left\{\frac{\psi_1}{2\psi_2}, -\left(\frac{p(s,x)-x}{x}\right) + \Lambda\right\}
\end{aligned}$$

Observe that in this case, as long as the net population growth rate is not negative enough, that the city planner sets the leaving rate at the point of maximum benefits $\frac{\psi_1}{2\psi_2}$ and lets the arrival rate make all the adjustment.

2) Suppose that $\psi_1 = \psi_2 = 0$.

Then, the FOC's become

$$-2H\frac{a(s,x)}{x} + \lambda\xi(s,x) - \lambda\eta \leq 0 \,(\text{with } = \text{ if } a(s,x) > 0)$$

$$-\lambda\xi(s,x) + \lambda\eta \leq 0 \,(\text{with } = \text{ if } l(s,x) > 0)$$

i) Suppose that $\frac{p(s,x)-x}{x} - \Lambda > 0$. Then,

$$\frac{a(s,x)}{x} = \frac{p(s,x)-x}{x} - \Lambda + \frac{l(s,x)}{x} > 0$$

Hence,

$$\lambda\xi(s,x) - \lambda\eta = 2H\frac{a(s,x)}{x} > 0$$

It follows that $l(s,x) = 0$.

Thus,

$$\begin{aligned}
\frac{l(s,x)}{x} &= 0 \\
\frac{a(s,x)}{x} &= \frac{p(s,x)-x}{x} - \Lambda > 0
\end{aligned}$$

ii) Suppose that $\frac{p(s,x)-x}{x} - \Lambda < 0$.

Then,

$$\frac{l(s,x)}{x} = \frac{a(s,x)}{x} - \left(\frac{p(s,x)-x}{x} - \Lambda\right) > 0$$

Hence,

$$-\lambda\xi(s,x) + \lambda\eta = 0$$

Assume that $a(s,x)>0$. Then,

$$\lambda\xi\left(s,x\right)-\lambda\eta=2H\frac{a\left(s,x\right)}{x}>0$$

A contradiction.

It follows that $a\left(s,x\right)=0$.

Thus,

$$\begin{aligned}\frac{l\left(s,x\right)}{x} &= -\left(\frac{p\left(s,x\right)-x}{x}-\Lambda\right)>0\\ \frac{a\left(s,x\right)}{x} &= 0\end{aligned}$$

(iii) Suppose that $\frac{p(s,x)-x}{x}-\Lambda=0$.

Hence,

$$\frac{a\left(s,x\right)}{x}=\frac{l\left(s,x\right)}{x}$$

Suppose that

$$\frac{a\left(s,x\right)}{x}=\frac{l\left(s,x\right)}{x}>0$$

Then,

$$\lambda\xi\left(s,x\right)-\lambda\eta=2H\frac{a\left(s,x\right)}{x}>0$$

and,

$$-\lambda\xi\left(s,x\right)+\lambda\eta=0$$

A contradiction.

It follows that

$$\frac{a\left(s,x\right)}{x}=\frac{l\left(s,x\right)}{x}=0$$

iv) From i), ii) and iii) we have that

$$\begin{aligned}\frac{l\left(s,x\right)}{x} &= \max\left\{-\left(\frac{p\left(s,x\right)-x}{x}-\Lambda\right),0\right\}\\ \frac{a\left(s,x\right)}{x} &= \max\left\{\frac{p\left(s,x\right)-x}{x}-\Lambda,0\right\}\end{aligned}$$

Observe that in this case the city planner always goes to a corner: when net population growth is positive he sets leaving rates to zero, and when net population growth is negative he sets arrival rates to zero. Since $\psi_1=\psi_2=0$ and $H>0$ it follows that the city planner faces adjustment costs to increase population but not to decrease it.

1.3 Proof that moving costs imply quadratic adjustment costs in total population

The derivations assume that $\sum_{s,x} x\mu(s,x) = 1$,i.e. that aggregate population is equal to one.

Assume that $a(s,x) > 0$ and $l(s,x) > 0$ almost everywhere (which again, is the empirically relevant case). Then, the first order conditions become the following:

$$-\tau + \sum_{s,x} \lambda\xi(s,x) x\mu(s,x) - \lambda\eta \leq 0 \,(\text{with } = \text{ if } \Lambda > 0)$$

$$\psi_1 - 2\psi_2 \frac{l(s,x)}{x} = 2H\frac{a(s,x)}{x}$$

$$\psi_1 - 2\psi_2 \frac{l(s,x)}{x} = \lambda\xi(s,x) - \lambda\eta$$

$$\xi(s,x) = s\theta p(s,x)^{\theta-1}$$

$$C = \sum_{s,x} sp(s,x)^{\theta} \mu(s,x)$$

$$p(s,x) = x + a(s,x) + \Lambda x - l(s,x)$$

$$\sum_{s,x} [a(s,x) + \Lambda x] \mu(s,x) = \sum_{s,x} l(s,x) \mu(s,x)$$

Directed arrival and leaving rates are then given by:

$$\frac{a(s,x)}{x} = \frac{\psi_1}{2H} - \frac{\psi_2}{H}\frac{l(s,x)}{x}$$

$$\begin{aligned}
\frac{l(s,x)}{x} &= \frac{a(s,x)}{x} + \Lambda - \frac{p(s,x)-x}{x} \\
&= \frac{\psi_1}{2H} - \frac{\psi_2}{H}\frac{l(s,x)}{x} + \Lambda - \frac{p(s,x)-x}{x}
\end{aligned}$$

Hence

$$\frac{l(s,x)}{x} = \frac{\psi_1}{2(H+\psi_2)} + \frac{H}{H+\psi_2}\Lambda - \frac{H}{H+\psi_2}\left(\frac{p(s,x)-x}{x}\right)$$

$$\frac{a(s,x)}{x} = \frac{\psi_1}{2H} - \frac{\psi_2}{H}\frac{\psi_1}{2(H+\psi_2)} - \frac{\psi_2}{H+\psi_2}\Lambda + \frac{\psi_2}{H+\psi_2}\left(\frac{p(s,x)-x}{x}\right)$$

Observe that

$$\begin{aligned}
\frac{\psi_1}{2H} - \frac{\psi_2}{H}\frac{\psi_1}{2(H+\psi_2)} &= \frac{\psi_1 H + \psi_1\psi_2 - \psi_1\psi_2}{2H(H+\psi_2)} \\
&= \frac{\psi_1}{2(H+\psi_2)}
\end{aligned}$$

Hence

$$\frac{a(s,x)}{x} = \frac{\psi_1}{2(H+\psi_2)} - \frac{\psi_2}{H+\psi_2}\Lambda + \frac{\psi_2}{H+\psi_2}\left(\frac{p(s,x)-x}{x}\right)$$
$$\frac{l(s,x)}{x} = \frac{\psi_1}{2(H+\psi_2)} + \frac{H}{H+\psi_2}\Lambda - \frac{H}{H+\psi_2}\left(\frac{p(s,x)-x}{x}\right)$$

Going back to the objective function, we have that

$$\begin{aligned}
& -H\left[\frac{a(s,x)}{x}\right]^2 x + \left[\psi_1\frac{l(s,x)}{x} - \psi_2\left(\frac{l(s,x)}{x}\right)^2\right] x \\
= & -H\left[\frac{\psi_1}{2(H+\psi_2)} - \frac{\psi_2}{H+\psi_2}\Lambda + \frac{\psi_2}{H+\psi_2}\left(\frac{p-x}{x}\right)\right]^2 x \\
& +\psi_1\left[\frac{\psi_1}{2(H+\psi_2)} + \frac{H}{H+\psi_2}\Lambda - \frac{H}{H+\psi_2}\left(\frac{p-x}{x}\right)\right] x \\
& -\psi_2\left[\frac{\psi_1}{2(H+\psi_2)} + \frac{H}{H+\psi_2}\Lambda - \frac{H}{H+\psi_2}\left(\frac{p-x}{x}\right)\right]^2 x \\
= & -Hx\left[\frac{\psi_1}{2(H+\psi_2)} - \frac{\psi_2}{H+\psi_2}\Lambda\right]^2 \\
& -Hx2\left[\frac{\psi_1}{2(H+\psi_2)} - \frac{\psi_2}{H+\psi_2}\Lambda\right]\left[\frac{\psi_2}{H+\psi_2}\left(\frac{p-x}{x}\right)\right] \\
& -Hx\left[\frac{\psi_2}{H+\psi_2}\left(\frac{p-x}{x}\right)\right]^2 \\
& +\psi_1 x\left[\frac{\psi_1}{2(H+\psi_2)} + \frac{H}{H+\psi_2}\Lambda\right] \\
& -\psi_1 x\frac{H}{H+\psi_2}\left(\frac{p-x}{x}\right) \\
& -\psi_2 x\left[\frac{\psi_1}{2(H+\psi_2)} + \frac{H}{H+\psi_2}\Lambda\right]^2 \\
& +\psi_2 x2\left[\frac{\psi_1}{2(H+\psi_2)} + \frac{H}{H+\psi_2}\Lambda\right]\left[\frac{H}{H+\psi_2}\left(\frac{p-x}{x}\right)\right] \\
& -\psi_2 x\left[\frac{H}{H+\psi_2}\left(\frac{p-x}{x}\right)\right]^2
\end{aligned}$$

$$\begin{aligned}
= & -\Phi(\Lambda) - \Gamma(\Lambda)(p-x) - Hx\left[\frac{\psi_2}{H+\psi_2}\left(\frac{p-x}{x}\right)\right]^2 - \psi_2 x\left[\frac{H}{H+\psi_2}\left(\frac{p-x}{x}\right)\right]^2 \\
= & -\Phi(\Lambda) - \Gamma(\Lambda)(p-x) - \left(\frac{p-x}{x}\right)^2\left\{Hx\left[\frac{\psi_2}{H+\psi_2}\right]^2 + \psi_2 x\left[\frac{H}{H+\psi_2}\right]^2\right\} \\
= & -\Phi(\Lambda) - \Gamma(\Lambda)(p-x) - \frac{H\psi_2}{H+\psi_2}\left(\frac{p-x}{x}\right)^2 x
\end{aligned}$$

Thus, the social planner's problem can be written as:

$$\max \left\{ \ln(C) - \sum_{s,x} \left[\Phi(\Lambda) + \Gamma(\Lambda)(p - x) + \frac{H\psi_2}{H + \psi_2} \left(\frac{p(s,x) - x}{x} \right)^2 x \right] \mu(s,x) - \tau\Lambda \right\}$$

subject to:

$$\sum_{s,x} p(s,x)\,\mu(s,x) \leq \sum_{s,x} x\mu(s,x)$$

$$C \leq \sum_{s,x} sp(s,x)^{\theta} \mu(s,x)$$

Using the second constraint, we then have that the social planner's problem can be written as:

$$\max \left\{ \ln\left[\sum_{s,x} sp(s,x)^{\theta} \mu(s,x) \right] - \sum_{s,x} \left[\Phi(\Lambda) + \frac{H\psi_2}{H + \psi_2} \left(\frac{p(s,x) - x}{x} \right)^2 x \right] \mu(s,x) - \tau\Lambda \right\}$$

subject to:

$$\sum_{s,x} p(s,x)\,\mu(s,x) \leq \sum_{s,x} x\mu(s,x)$$

1.4 Competitive equilibrium

There is a representative household with a continuum of members. Its members are distributed across city types (s, x) according to the measure μ.

1.4.1 Household problem

$$\max \left\{ \ln C - \sum_{s,x} \left[H \left(\frac{a(s,x)}{x} \right)^2 x + \left(-\psi_1 \frac{l(s,x)}{x} + \psi_2 \left(\frac{l(s,x)}{x} \right)^2 \right) x \right] \mu(s,x) - \tau\Lambda \right\}$$

subject to:

$$C + \sum_{s,x} q(s,x)\, m(s,x) \mu(s,x) \leq \sum_{s,x} q(s,x)\, a(s,x) \mu(s,x) + \sum_{s,x} w(s,x)\, p(s,x) \mu(s,x) + \Pi$$

$$p(s,x) \leq x + m(s,x) + \Lambda x - l(s,x)$$

$$\sum_{s,x} \left[m(s,x) + \Lambda x \right] \mu(s,x) \leq \sum_{s,x} l(s,x)\, \mu(s,x)$$

1.4.2 Firms' problem in city of type (s, x)

$$\max \left\{ sn(s,x)^{\theta} - w(s,x) \right\}$$

1.4.3 Market clearing conditions

$$n(s,x) = p(s,x)$$
$$m(s,x) = a(s,x)$$

for each type of city (s,x), and

$$C = \sum_{s,x} sn(s,x)^{\theta} \mu(s,x)$$

1.4.4 First order conditions

Households:

$$\frac{1}{C} = \lambda$$
$$-\tau + \sum_{s,x} \lambda\xi(s,x)\, x\mu(s,x) - \lambda\eta \leq 0 \,(\text{with } = \text{ if } \Lambda > 0)$$
$$-2H\frac{a(s,x)}{x} + \lambda q(s,x) \leq 0 \,(\text{with } = \text{ if } a(s,x) > 0)$$
$$-\lambda q(s,x) + \lambda\xi(s,x) - \lambda\eta \leq 0 \,(\text{with } = \text{ if } m(s,x) > 0)$$
$$\left[\psi_1 - 2\psi_2\frac{l(s,x)}{x}\right] - \lambda\xi(s,x) + \lambda\eta \leq 0 \,(\text{with } = \text{ if } l(s,x) > 0)$$
$$\lambda w(s,x) = \lambda\xi(s,x)$$

Firms:

$$w(s,x) = s\theta n(s,x)^{\theta-1}$$

Since at equilibrium $a(s,x) = m(s,x)$, we have that

(i) if $a(s,x) > 0$

$$\lambda q(s,x) = 2H\frac{a(s,x)}{x} = \lambda\xi(s,x) - \lambda\eta > 0$$

(ii) if $a(s,x) = 0$

$$0 \leq \lambda q(s,x) \leq 2H\frac{a(s,x)}{x} = 0$$

i.e. $q(s,x) = 0$, and

$$\xi(s,x) - \eta \leq q(s,x) = 0$$

We then have that

$$q(s,x) = \max\{0, \xi(s,x) - \eta\}$$
$$w(s,x) = s\theta n(s,x)^{\theta-1}$$

With this characterization of equilibrium prices it is straightforward to verify that the welfare theorems hold.

1.5 Competitive equilibrium with guided trips produced both at home and the market

There is a representative household with a continuum of members. Its members are distributed across city types (s, x) according to the measure μ.

1.5.1 Household problem

$$\max\left\{\ln C-\sum_{s,x}\left[H\left(\frac{a_m\left(s,x\right)+a_h\left(s,x\right)}{x}\right)^2 x+\left(-\psi_1\frac{l\left(s,x\right)}{x}+\psi_2\left(\frac{l\left(s,x\right)}{x}\right)^2\right)x\right]\mu\left(s,x\right)-\tau\Lambda\right\}$$

subject to:

$$C+\sum_{s,x} q\left(s,x\right)m(s,x)\mu\left(s,x\right)\leq\sum_{s,x} q\left(s,x\right)a_m(s,x)\mu\left(s,x\right)+\sum_{s,x} w\left(s,x\right)p(s,x)\mu\left(s,x\right)+\Pi$$

$$p\left(s,x\right)\leq x+m\left(s,x\right)+a_h\left(s,x\right)+\Lambda x-l\left(s,x\right) \tag{1}$$

$$\sum_{s,x}\left[m\left(s,x\right)+a_h\left(s,x\right)+\Lambda x\right]\mu\left(s,x\right)\leq\sum_{s,x} l\left(s,x\right)\mu\left(s,x\right) \tag{2}$$

1.5.2 Firms' problem in city of type (s, x)

$$\max\left\{sn\left(s,x\right)^\theta-w\left(s,x\right)\right\}$$

1.5.3 Market clearing conditions

$$n\left(s,x\right)=p\left(s,x\right) \tag{3}$$

$$m(s,x)=a_m\left(s,x\right) \tag{4}$$

for each type of city (s, x), and

$$C=\sum_{s,x} sn\left(s,x\right)^\theta\mu\left(s,x\right) \tag{5}$$

1.5.4 First order conditions

Households:

$$\frac{1}{C}=\lambda \tag{6}$$

$$-\tau+\sum_{s,x}\lambda\xi\left(s,x\right)x\mu\left(s,x\right)-\lambda\eta\leq 0\,(\text{with }=\text{ if }\Lambda>0) \tag{7}$$

$$-2H\frac{a_m(s,x)+a_h(s,x)}{x}+\lambda q(s,x)\leq 0\,(\text{with } = \text{ if } a_m(s,x)>0) \tag{8}$$

$$-2H\frac{a_m(s,x)+a_h(s,x)}{x}+\lambda\xi(s,x)-\lambda\eta\leq 0\,(\text{with } = \text{ if } a_h(s,x)>0) \tag{9}$$

$$-\lambda q(s,x)+\lambda\xi(s,x)-\lambda\eta\leq 0\,(\text{with } = \text{ if } m(s,x)>0) \tag{10}$$

$$\left[\psi_1-2\psi_2\frac{l(s,x)}{x}\right]-\lambda\xi(s,x)+\lambda\eta\leq 0\,(\text{with } = \text{ if } l(s,x)>0) \tag{11}$$

$$\lambda w(s,x)=\lambda\xi(s,x) \tag{12}$$

Firms:

$$w(s,x)=s\theta n(s,x)^{\theta-1} \tag{13}$$

Using that $m(s,x)=a_m(s,x)$, from equations (1)-(13) we verify that a competitive equilibrium only determines the total number of guided trips into a city $a_m(s,x)+a_h(s,x)$: the composition of these guided trips between market activities $a_m(s,x)$ and home activities $a_h(s,x)$ is left undetermined.

(i) In an equilibrium in which $m(s,x)=a_m(s,x)>0$ we have that

$$\lambda q(s,x)=2H\frac{a_m(s,x)+a_h(s,x)}{x}=\lambda\xi(s,x)-\lambda\eta>0$$

(ii) In an equilibrium in which $m(s,x)=a_m(s,x)=0$ and $a_h(s,x)>0$ we have that

$$\lambda q(s,x)\leq 2H\frac{a_m(s,x)+a_h(s,x)}{x}=\lambda\xi(s,x)-\lambda\eta$$

(iii) In an equilibrium in which $m(s,x)=a_m(s,x)=a_h(s,x)=0$ we have that

$$0\leq\lambda q(s,x)\leq 2H\frac{a_m(s,x)+a_h(s,x)}{x}=0$$

i.e. $q(s,x)=0$, and

$$\xi(s,x)-\eta\leq q(s,x)=0$$

We then have that

$$q(s,x)=\max\{0,\xi(s,x)-\eta\}$$
$$w(s,x)=s\theta n(s,x)^{\theta-1}$$

With this characterization of equilibrium prices it is straightforward to verify that the welfare theorems hold.

1.6 Adjustment rules in the general case

First order conditions:

$$\psi_1 - 2\psi_2 \frac{l}{x} \leq 2H\frac{a}{x}, \text{ (with equality if } \frac{a}{x} > 0 \text{ and } \frac{l}{x} > 0\text{)}$$

$$\frac{p}{x} = 1 + \frac{a}{x} + \Lambda - \frac{l}{x}$$

Define

$$\frac{A}{x} = \frac{a}{x} + \Lambda$$

Take $\frac{p}{x}$ as given.

1) Check case $\frac{a}{x} = 0$.

Define

$$\left(\frac{l}{x}\right)^* = 1 + \Lambda - \frac{p}{x}$$

If

$$\psi_1 - 2\psi_2 \left(\frac{l}{x}\right)^* \leq 0$$

Then,

$$\begin{aligned} \frac{A}{x} &= \Lambda \\ \frac{l}{x} &= \left(\frac{l}{x}\right)^* \end{aligned}$$

2) Else, check case $\frac{l}{x} = 0$.

Define

$$\left(\frac{A}{x}\right)^* = \frac{p}{x} - 1$$

If

$$\psi_1 \leq 2H\left[\left(\frac{A}{x}\right)^* - \Lambda\right]$$

Then,

$$\begin{aligned} \frac{A}{x} &= \left(\frac{A}{x}\right)^* \\ \frac{l}{x} &= 0 \end{aligned}$$

3) Else, consider case $\frac{a}{x} > 0$ and $\frac{l}{x} > 0$.

$$\frac{\psi_1}{2H} - \frac{\psi_2}{H}\frac{l}{x} = \frac{a}{x}$$
$$\frac{p}{x} = 1 + \frac{a}{x} + \Lambda - \frac{l}{x}$$

Hence,

$$\begin{aligned}
\frac{p}{x} &= 1 + \frac{\psi_1}{2H} - \frac{\psi_2}{H}\frac{l}{x} + \Lambda - \frac{l}{x} \\
&= 1 + \frac{\psi_1}{2H} - \frac{\psi_2}{H}\frac{l}{x} + \Lambda - \frac{l}{x} \\
&= 1 + \frac{\psi_1}{2H} - \left(\frac{\psi_2 + H}{H}\right)\frac{l}{x} + \Lambda
\end{aligned}$$

That is,

$$\begin{aligned}
\frac{l}{x} &= \left(\frac{H}{\psi_2 + H}\right)\left[1 + \frac{\psi_1}{2H} + \Lambda - \frac{p}{x}\right] \\
\frac{A}{x} &= \frac{p}{x} - 1 + \frac{l}{x}
\end{aligned}$$

Observe that $\frac{p}{x}$ below which $\frac{A}{x} = \Lambda$ is

$$\left(\frac{p}{x}\right)^* = 1 + \Lambda - \frac{\psi_1}{2\psi_2}$$

2 Dynamic Economy

2.1 Recursive formulation of economy-wide social planner problem

$$J(K,\mu) = \max\left\{\ln C - \tau\Lambda - \int \phi\,(n_y + n_h)^{\pi}\, p^{1-\pi} d\mu + \int A \ln\left(\frac{h^{\varsigma} b_r^{1-\varsigma}}{p}\right) p d\mu \right.$$
$$\left. - \int H\left(\frac{a}{x}\right)^2 x d\mu + \int \left[\psi_1 \frac{l}{x} - \psi_2\left(\frac{l}{x}\right)^2\right] x d\mu + \beta J(K', \mu')\right\}$$

subject to

$$p(h,x,s,s_{-1}) = x + a(h,x,s,s_{-1}) + \Lambda x - l$$
$$a(h,x,s,s_{-1}) \geq 0$$
$$l(h,x,s,s_{-1}) \geq 0$$
$$n_y(h,x,s,s_{-1}) + n_h(h,x,s,s_{-1}) \leq p(h,x,s,s_{-1})$$
$$b_r(h,x,s,s_{-1}) + b_h(h,x,s,s_{-1}) = \bar{b}$$

$$\int a \; d\mu + \Lambda = \int l \; d\mu$$

$$\int \left[k_y + k_h\right] d\mu = K$$

$$C + I = \left\{ \int \left[s n_y^{\theta} k_y^{\gamma} \right]^{\chi} d\mu \right\}^{\frac{1}{\chi}}$$

$$K' = (1 - \delta_k) + I$$

$$\mu' \left(H \times X, s', s\right) = \int_{B(H \times X, s, s_{-1})} Q\left(s'; s, s_{-1}\right) d\mu$$

$$\begin{aligned} B\left(H \times X, s, s_{-1}\right) \;\; = \;\; & \{ (h, x, s, s_{-1}) : p\left(h, x, s, s_{-1}\right) \in X \\ & \text{and } \left(1 - \delta_h\right) h + n_h^{\alpha}\left(h, x, s, s_{-1}\right) k_h^{\lambda}\left(h, x, s, s_{-1}\right) b_h^{1-\alpha-\lambda}\left(h, x, s, s_{-1}\right) \in H \big\} \end{aligned}$$

Definition: A steady state is an aggregate state (K^{*}, μ^{*}) that is time invariant under the optimal decision rules.

Section 2.5 provides a characterization of this steady state in terms of a much simpler dynamic programming problem and certain side conditions. In order to arrive to this characterization it will be convenient to work with a sequential formulation to the economy-wide social planner problem. The next section provides such formulation.

2.2 Economy-wide social planner problem in sequential form

Define

$$\mu_t\left(s^t, h_0, x_0, s_0, s_{-1}\right) = Q\left(s_t; s_{t-1}, s_{t-2}\right) ... Q(s_2; s_1, s_0) Q(s_1; s_0, s_{-1}) \mu_0\left(h_0, x_0, s_0, s_{-1}\right)$$

Then, the economy-wide social planner problem is the following:

$$\begin{aligned} \max \sum_{t=0}^{\infty} \beta^t \Bigg\{ \ln C_t - \sum_{s^t} \Big[& \phi \left(n_{y,t}\left(s^t, h_0, x_0, s_0, s_{-1}\right) + n_{h,t}\left(s^t, h_0, x_0, s_0, s_{-1}\right) \right)^{\pi} p_t\left(s^t, h_0, x_0, s_0, s_{-1}\right)^{1-\pi} \\ & + A \ln \left(\frac{h_t\left(s^t, h_0, x_0, s_0, s_{-1}\right)^{\varsigma} b_{r,t}\left(s^t, h_0, x_0, s_0, s_{-1}\right)^{1-\varsigma}}{p_t\left(s^t, h_0, x_0, s_0, s_{-1}\right)} \right) p_t\left(s^t, h_0, x_0, s_0, s_{-1}\right) \\ & - \tau \Lambda_t p_{t-1}\left(s^{t-1}, h_0, x_0, s_0, s_{-1}\right) \\ & - H \left(\frac{a_t\left(s^t, h_0, x_0, s_0, s_{-1}\right)}{p_{t-1}\left(s^{t-1}, h_0, x_0, s_0, s_{-1}\right)} \right)^2 p_{t-1}\left(s^{t-1}, h_0, x_0, s_0, s_{-1}\right) \\ & + \left(\psi_1 \frac{l_t\left(s^t, h_0, x_0, s_0, s_{-1}\right)}{p_{t-1}\left(s^{t-1}, h_0, x_0, s_0, s_{-1}\right)} - \psi_2 \left(\frac{l_t\left(s^t, h_0, x_0, s_0, s_{-1}\right)}{p_{t-1}\left(s^{t-1}, h_0, x_0, s_0, s_{-1}\right)} \right)^2 \right) p_{t-1}\left(s^{t-1}, h_0, x_0, s_0, s_{-1}\right) \end{aligned}$$

$$\times\mu_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)\big]\Big\}$$

subject to:

$$p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)=p_{t-1}\left(s^{t-1},h_0,x_0,s_0,s_{-1}\right)+a_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)$$
$$+\Lambda_t p_{t-1}\left(s^{t-1},h_0,x_0,s_0,s_{-1}\right)-l_t\left(s^t,h_0,x_0,s_0,s_{-1}\right) \quad (14)$$
$$a_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)\geq 0 \quad (15)$$
$$l_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)\geq 0 \quad (16)$$
$$n_{y,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)+n_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)\leq p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right) \quad (17)$$
$$b_{r,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)+b_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)=\bar{b} \quad (18)$$

$$h_{t+1}\left(s^{t+1},h_0,x_0,s_0,s_{-1}\right)=\left(1-\delta_h\right)h_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)$$
$$+n_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{\alpha}k_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{\lambda}b_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{1-\alpha-\lambda} \quad (19)$$

$$\sum a_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)\mu_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)+\Lambda_t$$
$$=\sum l_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)\mu_t\left(s^t,h_0,x_0,s_0,s_{-1}\right) \quad (20)$$

$$\sum\left[k_{y,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)+k_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)\right]\mu_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)=K_t \quad (21)$$

$$C_t+K_{t+1}-\left(1-\delta_k\right)K_t$$
$$=\left\{\sum\left[s_t n_{y,t}^{\theta}\left(s^t,h_0,x_0,s_0,s_{-1}\right)k_{y,t}^{\gamma}\left(s^t,h_0,x_0,s_0,s_{-1}\right)\right]^{\chi}\mu_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)\right\}^{\frac{1}{\chi}} \quad (22)$$

with (K_0,μ_0) given.

FOC's:

$$\varphi_t=\frac{1}{C_t} \quad (23)$$

$$\sum\varphi_t\xi_t(s^t,h_0,x_0,s_0,s_{-1})p_{t-1}\left(s^{t-1},h_0,x_0,s_0,s_{-1}\right)\mu_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)-\varphi_t\eta_t\leq\tau,\ \ (=0\text{ if }\Lambda_t>0) \quad (24)$$

$$\varphi_t\xi_t(s^t,h_0,x_0,s_0,s_{-1})-\varphi_t\eta_t\leq H2\left(\frac{a_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)}{p_{t-1}\left(s^{t-1},h_0,x_0,s_0,s_{-1}\right)}\right),\ \ \left(=0\text{ if }a_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)>0\right) \quad (25)$$

$$\psi_1-2\psi_2\left(\frac{l_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)}{p_{t-1}\left(s^{t-1},h_0,x_0,s_0,s_{-1}\right)}\right)\leq\varphi_t\xi_t(s^t,h_0,x_0,s_0,s_{-1})-\varphi_t\eta_t 0,\ \ \left(=0\text{ if }l_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)>0\right) \quad (26)$$

$$\varphi_t\xi_t(s^t,h_0,x_0,s_0,s_{-1})=$$

$$-\phi\left(n_{y,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)+n_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)\right)^{\pi}(1-\pi)\,p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{-\pi}$$

$$+A\ln\left(\frac{h_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{\varsigma}b_{r,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{1-\varsigma}}{p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)}\right)-A$$

$$+\beta\sum_{s_{t+1}}H\left(\frac{a_{t+1}\left(s^{t+1},h_0,x_0,s_0,s_{-1}\right)}{p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)}\right)^2 Q(s_{t+1};s_t,s_{t-1})$$

$$+\beta\sum_{s_{t+1}}\left[\psi_1\frac{l_{t+1}\left(s^{t+1},h_0,x_0,s_0,s_{-1}\right)}{p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)}-\psi_2\left(\frac{l_{t+1}\left(s^{t+1},h_0,x_0,s_0,s_{-1}\right)}{p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)}\right)^2\right]Q(s_{t+1};s_t,s_{t-1})$$

$$-\beta\sum_{s_{t+1}}\left[\psi_1\frac{l_{t+1}\left(s^{t+1},h_0,x_0,s_0,s_{-1}\right)}{p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)}-2\psi_2\left(\frac{l_{t+1}\left(s^{t+1},h_0,x_0,s_0,s_{-1}\right)}{p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)}\right)^2\right]Q(s_{t+1};s_t,s_{t-1})$$

$$+\sum_{s_{t+1}}\beta\varphi_{t+1}\xi_{t+1}(s^{t+1},h_0,x_0,s_0,s_{-1})\,(1+\Lambda_{t+1})\,Q(s_{t+1};s_t,s_{t-1})$$

$$+\varphi_t\left[w_t(s^t,h_0,x_0,s_0,s_{-1})\right.$$

$$\left.-\frac{1}{\varphi_t}\phi\pi\left[n_{y,t}(s^t,h_0,x_0,s_0,s_{-1})+n_{h,t}(s^t,h_0,x_0,s_0,s_{-1})\right]^{\pi-1}p_t(s^t,h_0,x_0,s_0,s_{-1})^{1-\pi}\right]$$

$$-\Lambda_{t+1}\beta\varphi_{t+1}\eta_{t+1} \tag{27}$$

$$w_t(s^t,h_0,x_0,s_0,s_{-1})\geq\frac{1}{\varphi_t}\phi\pi\left[n_{y,t}(s^t,h_0,x_0,s_0,s_{-1})+n_{h,t}(s^t,h_0,x_0,s_0,s_{-1})\right]^{\pi-1}p_t(s^t,h_0,x_0,s_0,s_{-1})^{1-\pi} \tag{28}$$

$$\left[w_t(s^t,h_0,x_0,s_0,s_{-1})-\frac{1}{\varphi_t}\phi\pi\left[n_{y,t}(s^t,h_0,x_0,s_0,s_{-1})+n_{h,t}(s^t,h_0,x_0,s_0,s_{-1})\right]^{\pi-1}p_t(s^t,h_0,x_0,s_0,s_{-1})^{1-\pi}\right]$$
$$\times\left[p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)-n_{y,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)-n_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)\right]=0 \tag{29}$$

$$\begin{aligned}w_t(s^t,h_0,x_0,s_0,s_{-1}) &= \left\{\sum\left[s_t n_{y,t}^{\theta}\left(s^t,h_0,x_0,s_0,s_{-1}\right)k_{y,t}^{\gamma}\left(s^t,h_0,x_0,s_0,s_{-1}\right)\right]^{\chi}\mu_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)\right\}^{\frac{1}{\chi}-1}\\ &\quad\left[s_t n_{y,t}^{\theta}\left(s^t,h_0,x_0,s_0,s_{-1}\right)k_{y,t}^{\gamma}\left(s^t,h_0,x_0,s_0,s_{-1}\right)\right]^{\chi-1}\\ &\quad s_t\theta n_{y,t}^{\theta-1}\left(s^t,h_0,x_0,s_0,s_{-1}\right)k_{y,t}^{\gamma}\left(s^t,h_0,x_0,s_0,s_{-1}\right)\end{aligned} \tag{30}$$

$$\begin{aligned}&w_t(s^t,h_0,x_0,s_0,s_{-1})= \\ &\vartheta_t(s^t,h_0,x_0,s_0,s_{-1})\alpha n_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{\alpha-1}k_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{\lambda}b_h\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{1-\alpha-\lambda}\end{aligned} \tag{31}$$

$$\begin{aligned}\varphi_t\vartheta_t(s^t,h_0,x_0,s_0,s_{-1}) &= \sum_{s_{t+1}}\beta\varphi_{t+1}\vartheta_{t+1}(s^{t+1},h_0,x_0,s_0,s_{-1})\,(1-\delta_h)\,Q(s_{t+1};s_t,s_{t-1})\\ &\quad+\sum_{s_{t+1}}\beta A\varsigma\frac{p_{t+1}\left(s^{t=1},h_0,x_0,s_0,s_{-1}\right)}{h_{t+1}\left(s^{t+1},h_0,x_0,s_0,s_{-1}\right)}Q(s_{t+1};s_t,s_{t-1})\end{aligned} \tag{32}$$

$$\varphi_t r_{bt}(s^t, h_0, x_0, s_0, s_{-1}) = A\left(1-\varsigma\right)\frac{p_t\left(s^t, h_0, x_0, s_0, s_{-1}\right)}{b_{r,t}\left(s^t, h_0, x_0, s_0, s_{-1}\right)} \tag{33}$$

$$r_{bt}(s^t, h_0, x_0, s_0, s_{-1}) = \vartheta_t(s^t, h_0, x_0, s_0, s_{-1}) \times \tag{34}$$
$$n_{h,t}\left(s^t, h_0, x_0, s_0, s_{-1}\right)^{\alpha} k_{h,t}\left(s^t, h_0, x_0, s_0, s_{-1}\right)^{\lambda}\left(1-\alpha-\lambda\right) b_h\left(s^t, h_0, x_0, s_0, s_{-1}\right)^{-\alpha-\lambda}$$

$$r_{kt} = \vartheta_t(s^t, h_0, x_0, s_0, s_{-1}) n_{h,t}\left(s^t, h_0, x_0, s_0, s_{-1}\right)^{\alpha} \lambda k_{h,t}\left(s^t, h_0, x_0, s_0, s_{-1}\right)^{\lambda-1} b_h\left(s^t, h_0, x_0, s_0, s_{-1}\right)^{1-\alpha-\lambda} \tag{35}$$

$$\begin{aligned} r_{kt} &= \left\{\sum\left[s_t n_{y,t}^{\theta}\left(s^t, h_0, x_0, s_0, s_{-1}\right) k_{y,t}^{\gamma}\left(s^t, h_0, x_0, s_0, s_{-1}\right)\right]^{\chi} \mu_t\left(s^t, h_0, x_0, s_0, s_{-1}\right)\right\}^{\frac{1}{\chi}-1} && (36)\\ &\left[s_t n_{y,t}^{\theta}\left(s^t, h_0, x_0, s_0, s_{-1}\right) k_{y,t}^{\gamma}\left(s^t, h_0, x_0, s_0, s_{-1}\right)\right]^{\chi-1} \\ &s_t n_{y,t}^{\theta}\left(s^t, h_0, x_0, s_0, s_{-1}\right) \gamma k_{y,t}^{\gamma-1}\left(s^t, h_0, x_0, s_0, s_{-1}\right) && (37)\end{aligned}$$

$$-\varphi_t + \beta\varphi_{t+1}\left(1-\delta_k\right) + \beta\varphi_{t+1} r_{k,t+1} = 0 \tag{38}$$

2.3 City planner's problem in sequential form

Takes as given $\{Y_t, \varphi_t, \eta_t, r_{kt}, \Lambda_t\}_{t=0}^{\infty}$.

Define
$$\hat{\mu}\left(s^t\right) = Q\left(s_t; s_{t-1}, s_{t-2}\right)...Q(s_2; s_1, s_0)Q(s_1; s_0, s_{-1}).$$

Then, the city planner's problem is the following:

$$\begin{aligned} \max\sum_{t=0}^{\infty}\beta\sum_{s^t}&\left\{\varphi_t\frac{1}{\chi}Y_t^{1-\chi}\left[s_t n_{y,t}^{\theta}\left(s^t, h_0, x_0, s_0, s_{-1}\right) k_{y,t}^{\gamma}\left(s^t, h_0, x_0, s_0, s_{-1}\right)\right]^{\chi}\right. \\ &-\phi\left(n_{y,t}\left(s^t, h_0, x_0, s_0, s_{-1}\right) + n_{h,t}\left(s^t, h_0, x_0, s_0, s_{-1}\right)\right)^{\pi} p_t\left(s^t, h_0, x_0, s_0, s_{-1}\right)^{1-\pi} \\ &+A\ln\left(\frac{h_t\left(s^t, h_0, x_0, s_0, s_{-1}\right)^{\varsigma} b_{r,t}\left(s^t, h_0, x_0, s_0, s_{-1}\right)^{1-\varsigma}}{p_t\left(s^t, h_0, x_0, s_0, s_{-1}\right)}\right) p_t\left(s^t, h_0, x_0, s_0, s_{-1}\right) \\ &-\varphi_t r_{kt}\left[k_{y,t}\left(s^t, h_0, x_0, s_0, s_{-1}\right) + k_{h,t}\left(s^t, h_0, x_0, s_0, s_{-1}\right)\right] \\ &-\varphi_t\eta_t\left[a_t\left(s^t, h_0, x_0, s_0, s_{-1}\right) + \Lambda_t p_{t-1}\left(s^{t-1}, h_0, x_0, s_0, s_{-1}\right)\right] \\ &+\varphi_t\eta_t l_t\left(s^t, h_0, x_0, s_0, s_{-1}\right) \\ &-H\left(\frac{a_t\left(s^t, h_0, x_0, s_0, s_{-1}\right)}{p_{t-1}\left(s^{t-1}, h_0, x_0, s_0, s_{-1}\right)}\right)^2 p_{t-1}\left(s^{t-1}, h_0, x_0, s_0, s_{-1}\right) \\ &\left.+\left[\psi_1\frac{l_t\left(s^t, h_0, x_0, s_0, s_{-1}\right)}{p_{t-1}\left(s^{t-1}, h_0, x_0, s_0, s_{-1}\right)} - \psi_2\left(\frac{l_t\left(s^t, h_0, x_0, s_0, s_{-1}\right)}{p_{t-1}\left(s^{t-1}, h_0, x_0, s_0, s_{-1}\right)}\right)^2\right] p_{t-1}\left(s^{t-1}, h_0, x_0, s_0, s_{-1}\right)\right\}\hat{\mu}\left(s^t\right)\end{aligned}$$

subject to:

$$p_t\left(s^t, h_0, x_0, s_0, s_{-1}\right) = p_{t-1}\left(s^{t-1}, h_0, x_0, s_0, s_{-1}\right) + a_t\left(s^t, h_0, x_0, s_0, s_{-1}\right)$$

$$+\Lambda_t p_{t-1}\left(s^{t-1},h_0,x_0,s_0,s_{-1}\right)-l_t\left(s^t,h_0,x_0,s_0,s_{-1}\right) \tag{39}$$

$$a_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)\geq 0 \tag{40}$$

$$l_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)\geq 0 \tag{41}$$

$$n_{y,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)+n_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)\leq p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right) \tag{42}$$

$$b_{r,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)+b_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)=\bar{b} \tag{43}$$

$$h_{t+1}\left(s^{t+1},h_0,x_0,s_0,s_{-1}\right)=\left(1-\delta_h\right)h_t\left(s^t,h_0,x_0,s_0,s_{-1}\right) \tag{44}$$

$$+n_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{\alpha}k_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{\lambda}b_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{1-\alpha-\lambda} \tag{45}$$

with (h_0,x_0,s_0,s_{-1}) given.

FOC's:

$$\varphi_t\xi_t(s^t,h_0,x_0,s_0,s_{-1})-\varphi_t\eta_t\leq H2\left(\frac{a_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)}{p_{t-1}\left(s^{t-1},h_0,x_0,s_0,s_{-1}\right)}\right),\quad \left(=0\text{ if }a_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)>0\right) \tag{46}$$

$$\psi_1-2\psi_2\left(\frac{l_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)}{p_{t-1}\left(s^{t-1},h_0,x_0,s_0,s_{-1}\right)}\right)\leq\varphi_t\xi_t(s^t,h_0,x_0,s_0,s_{-1})-\varphi_t\eta_t,\quad \left(=0\text{ if }l_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)>0\right) \tag{47}$$

$$\begin{aligned}
&\varphi_t\xi_t(s^t,h_0,x_0,s_0,s_{-1})=\\
&-\phi\left(n_{y,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)+n_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)\right)^{\pi}(1-\pi)\,p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{-\pi}\\
&+A\ln\left(\frac{h_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{\varsigma}b_{r,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{1-\varsigma}}{p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)}\right)-A\\
&+\beta\sum_{s_{t+1}}H\left(\frac{a_{t+1}\left(s^{t+1},h_0,x_0,s_0,s_{-1}\right)}{p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)}\right)^2 Q(s_{t+1};s_t,s_{t-1})\\
&+\beta\sum_{s_{t+1}}\left[\psi_1\frac{l_{t+1}\left(s^{t+1},h_0,x_0,s_0,s_{-1}\right)}{p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)}-\psi_2\left(\frac{l_{t+1}\left(s^{t+1},h_0,x_0,s_0,s_{-1}\right)}{p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)}\right)^2\right]Q(s_{t+1};s_t,s_{t-1})\\
&-\beta\sum_{s_{t+1}}\left[\psi_1\frac{l_{t+1}\left(s^{t+1},h_0,x_0,s_0,s_{-1}\right)}{p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)}-2\psi_2\left(\frac{l_{t+1}\left(s^{t+1},h_0,x_0,s_0,s_{-1}\right)}{p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)}\right)^2\right]Q(s_{t+1};s_t,s_{t-1})\\
&+\sum_{s_{t+1}}\beta\varphi_{t+1}\xi_{t+1}(s^{t+1},h_0,x_0,s_0,s_{-1})\left(1+\Lambda_{t+1}\right)Q(s_{t+1};s_t,s_{t-1})\\
&+\varphi_t\Big[w_t(s^t,h_0,x_0,s_0,s_{-1})\\
&-\frac{1}{\varphi_t}\phi\pi\left[n_{y,t}(s^t,h_0,x_0,s_0,s_{-1})+n_{h,t}(s^t,h_0,x_0,s_0,s_{-1})\right]^{\pi-1}p_t(s^t,h_0,x_0,s_0,s_{-1})^{1-\pi}\Big]\\
&-\Lambda_{t+1}\beta\varphi_{t+1}\eta_{t+1}
\end{aligned} \tag{48}$$

$$w_t(s^t,h_0,x_0,s_0,s_{-1}) \geq \frac{1}{\varphi_t}\phi\pi\left[n_{y,t}(s^t,h_0,x_0,s_0,s_{-1})+n_{h,t}(s^t,h_0,x_0,s_0,s_{-1})\right]^{\pi-1}p_t(s^t,h_0,x_0,s_0,s_{-1})^{1-\pi}$$

$$\left[w_t(s^t,h_0,x_0,s_0,s_{-1})-\frac{1}{\varphi_t}\phi\pi\left[n_{y,t}(s^t,h_0,x_0,s_0,s_{-1})+n_{h,t}(s^t,h_0,x_0,s_0,s_{-1})\right]^{\pi-1}p_t(s^t,h_0,x_0,s_0,s_{-1})^{1-\pi}\right]$$
$$\times\left[p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)-n_{y,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)-n_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)\right]=0 \qquad (49)$$

$$\begin{aligned} w_t(s^t,h_0,x_0,s_0,s_{-1}) &= Y_t^{1-\chi} \\ &\left[s_t n_{y,t}^{\theta}\left(s^t,h_0,x_0,s_0,s_{-1}\right)k_{y,t}^{\gamma}\left(s^t,h_0,x_0,s_0,s_{-1}\right)\right]^{\chi-1} \\ &s_t\theta n_{y,t}^{\theta-1}\left(s^t,h_0,x_0,s_0,s_{-1}\right)k_{y,t}^{\gamma}\left(s^t,h_0,x_0,s_0,s_{-1}\right) \end{aligned} \qquad (50)$$

$$\begin{aligned} &w_t(s^t,h_0,x_0,s_0,s_{-1})= \qquad (51)\\ &\vartheta_t(s^t,h_0,x_0,s_0,s_{-1})\alpha n_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{\alpha-1}k_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{\lambda}b_h\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{1-\alpha-\lambda}\end{aligned}$$

$$\begin{aligned} \varphi_t\vartheta_t(s^t,h_0,x_0,s_0,s_{-1}) &= \sum_{s_{t+1}}\beta\varphi_{t+1}\vartheta_{t+1}(s^{t+1},h_0,x_0,s_0,s_{-1})\left(1-\delta_h\right)Q(s_{t+1};s_t,s_{t-1}) \\ &+\sum_{s_{t+1}}\beta A\varsigma\frac{p_{t+1}\left(s^{t=1},h_0,x_0,s_0,s_{-1}\right)}{h_{t+1}\left(s^{t+1},h_0,x_0,s_0,s_{-1}\right)}Q(s_{t+1};s_t,s_{t-1}) \end{aligned} \qquad (52)$$

$$\varphi_t r_{bt}(s^t,h_0,x_0,s_0,s_{-1})=A\left(1-\varsigma\right)\frac{p_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)}{b_{r,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)} \qquad (53)$$

$$\begin{aligned} &r_{bt}(s^t,h_0,x_0,s_0,s_{-1})=\vartheta_t(s^t,h_0,x_0,s_0,s_{-1})\times \qquad (54)\\ &n_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{\alpha}k_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{\lambda}\left(1-\alpha-\lambda\right)b_h\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{-\alpha-\lambda}\end{aligned}$$

$$r_{kt}=\vartheta_t(s^t,h_0,x_0,s_0,s_{-1})n_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{\alpha}\lambda k_{h,t}\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{\lambda-1}b_h\left(s^t,h_0,x_0,s_0,s_{-1}\right)^{1-\alpha-\lambda} \qquad (55)$$

$$\begin{aligned} r_{kt} &= Y_t^{1-\chi} \qquad (56)\\ &\left[s_t n_{y,t}^{\theta}\left(s^t,h_0,x_0,s_0,s_{-1}\right)k_{y,t}^{\gamma}\left(s^t,h_0,x_0,s_0,s_{-1}\right)\right]^{\chi-1} \\ &s_t n_{y,t}^{\theta}\left(s^t,h_0,x_0,s_0,s_{-1}\right)\gamma k_{y,t}^{\gamma-1}\left(s^t,h_0,x_0,s_0,s_{-1}\right) \qquad (57)\end{aligned}$$

2.4 Equivalence between economy-wide social planner's problem and city planner's problem

Proposition 1: Let $\{C_t, K_{t+1}, n_{yt}, n_{ht}, k_{yt}, k_{ht}, h_{t+1}, b_{rt}, b_{ht}, p_t, \Lambda_t, a_t, l_t, \varphi_t, \xi_t, \eta_t, w_t, \vartheta_t, r_{bt}, r_{kt}\}_{t=0}^{\infty}$ be the unique solution to the economy-wide social planner problem with initial state (K_0, μ_0).

Define

$$Y_t=\left\{\sum\left[s_t n_{y,t}^{\theta}\left(s^t,h_0,x_0,s_0,s_{-1}\right)k_{y,t}^{\gamma}\left(s^t,h_0,x_0,s_0,s_{-1}\right)\right]^{\chi}\mu_t\left(s^t,h_0,x_0,s_0,s_{-1}\right)\right\}^{\frac{1}{\chi}}. \qquad (58)$$

Then, for each initial state (h_0, x_0, s_0, s_{-1}), $\{n_{yt}, n_{ht}, k_{yt}, k_{ht}, h_t, b_{rt}, b_{ht}, p_t, a_t, l_t, \xi_t, w_t, \vartheta_t, r_{bt}\}_{t=0}^{\infty}$ is the unique solution to the city planner's problem that takes $\{Y_t, \varphi_t, \eta_t, r_{kt}, \Lambda_t\}_{t=0}^{\infty}$ as given.

Proof: It follows from the fact that equations (14)-(19),(25)-(36),(58) imply equations (39)-(45),(46)-(57).

Proposition 2: For each initial state (h_0, x_0, s_0, s_{-1}) let $\{n_{yt},\ n_{ht}, k_{yt}, k_{ht}, h_{t+1}, b_{rt}, b_{ht}, p_t, a_t, l_t, \xi_t, w_t, \vartheta_t, r_{bt}\}_{t=0}^{\infty}$ be the unique solution to the city planner's problem that takes $\{Y_t, \varphi_t, \eta_t, r_{kt}, \Lambda_t\}_{t=0}^{\infty}$ as given.

Define

$$K_t = \sum \left[k_{y,t}\left(s^t, h_0, x_0, s_0, s_{-1}\right) + k_{h,t}\left(s^t, h_0, x_0, s_0, s_{-1}\right)\right] \mu_t\left(s^t, h_0, x_0, s_0, s_{-1}\right), \tag{59}$$

$$C_t = Y_t - K_{t+1} + (1-\delta_k) K_t. \tag{60}$$

Suppose that

$$Y_t = \left\{\sum \left[s_t n_{y,t}^{\theta}\left(s^t, h_0, x_0, s_0, s_{-1}\right) k_{y,t}^{\gamma}\left(s^t, h_0, x_0, s_0, s_{-1}\right)\right]^{\chi} \mu_t\left(s^t, h_0, x_0, s_0, s_{-1}\right)\right\}^{\frac{1}{\chi}}, \tag{61}$$

$$\varphi_t = \frac{1}{C_t},$$

$$-\varphi_t + \beta\varphi_{t+1}(1-\delta_k) + \beta\varphi_{t+1} r_{k,t+1} = 0, \tag{62}$$

$$\sum \varphi_t \xi_t(s^t, h_0, x_0, s_0, s_{-1}) p_{t-1}\left(s^{t-1}, h_0, x_0, s_0, s_{-1}\right) \mu_t\left(s^t, h_0, x_0, s_0, s_{-1}\right) - \varphi_t \eta_t \leq \tau, (= 0 \text{ if } \Lambda_t > 0), \tag{63}$$

$$\begin{aligned} & \sum a_t\left(s^t, h_0, x_0, s_0, s_{-1}\right) \mu_t\left(s^t, h_0, x_0, s_0, s_{-1}\right) + \Lambda_t \\ = & \sum l_t\left(s^t, h_0, x_0, s_0, s_{-1}\right) \mu_t\left(s^t, h_0, x_0, s_0, s_{-1}\right). \end{aligned} \tag{64}$$

Then, $\{C_t, K_{t+1}, n_{yt}, n_{ht}, k_{yt}, k_{ht}, h_{t+1}, b_{rt}, b_{ht}, p_t, \Lambda_t, a_t, l_t, \varphi_t, \xi_t, \eta_t, w_t, \vartheta_t, r_{bt}, r_{kt}\}_{t=0}^{\infty}$ is the unique solution to the economy-wide social planner problem with initial state (K_0, μ_0).

Proof: It follows from the fact that equations (39)-(45),(46)-(57),(61)-(60) imply equations (14)-(22),(23)-(38).

2.5 Characterization of a steady state allocation

Consider the following recursive formulation to the city planner's problem that takes $(Y, \varphi, \eta, r_k, \Lambda)$ as given.

$$V(h, x, s, s_{-1}) = \max\left\{\varphi\frac{1}{\chi}Y^{1-\chi}\left[s n_y^{\theta} k_y^{\gamma}\right]^{\chi} - \phi\left(n_y + n_h\right)^{\pi} p^{1-\pi} + A\ln\left(\frac{h^{\varsigma} b_r^{1-\varsigma}}{p}\right) p\right.$$
$$-\varphi r_k\left(k_y + k_h\right) - \varphi\eta\left(a + \Lambda x\right) + \varphi\eta l$$

$$-H\left(\frac{a}{x}\right)^2 x+\left[\psi_1\frac{l}{x}-\psi_2\left(\frac{l}{x}\right)^2\right]x+\beta\sum_{s'}V\left(h',x',s',s\right)Q\left(s';s,s_{-1}\right)\Bigg\}$$

subject to

$$p = x + a + \Lambda x - l$$

$$a \geq 0$$

$$l \geq 0$$

$$n_y + n_h \leq p$$

$$b_r + b_h = \bar{b}$$

$$h' = (1-\delta_h)\,h + n_h^\alpha k_h^\lambda b_h^{1-\alpha-\lambda}$$

$$x' = p$$

Proposition 3: Let $(n_y, n_h, k_y, k_h, h', b_r, b_h, p, a, l)$ be the optimal decision rules to the recursive city planner's problem that takes $(Y, \varphi, \eta, r_k, \Lambda)$ as given.

Let μ be the invariant distribution generated by the optimal decision rules (h', p) and the transition function Q.

Define

$$K = \int (k_y + k_h)\, d\mu \tag{65}$$

$$C = Y - \delta_k K$$

and

$$\xi\left(h,x,s,s_{-1}\right) =$$

$$\{CH2\left[\frac{a\left(h,x,s,s_{-1}\right)}{x}\right]+\eta,\text{ if } a\left(h,x,s,s_{-1}\right)>0, C\left[\psi_1-2\psi_2\left(\frac{l\left(h,x,s,s_{-1}\right)}{x}\right)\right]+\eta,\text{ otherwise.}$$

Suppose that

$$Y = \left\{\int \left[s n_y^\theta k_y^\gamma\right]^\chi d\mu\right\}^{\frac{1}{\chi}} \tag{66}$$

$$r_k = \frac{1}{\beta} - 1 + \delta_k \tag{67}$$

$$\int a d\mu + \Lambda = \int l d\mu \tag{68}$$

$$\int \frac{1}{C}\left[\xi-\eta\right] x d\mu \leq \tau, (= 0 \text{ if } \Lambda > 0)\,, \tag{69}$$

Then, $(C, K, n_y, n_h, k_y, k_h, h', b_r, b_h, p, \Lambda, a, l)$ describes a steady state allocation.

Proof: It follows from Proposition 2 and equations (46)-(47).